Mathematical Theory of Finite Elements
(A crash course for engineers)

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Abstract

We review fundamentals of Galerkin and conforming Finite Element (FE) methods using model diffusion-convection-reaction and linear elasticity problems. We discuss the possibility of different variational formulations leading to different energy spaces and corresponding conforming elements. The course is focusing on the famous inf-sup stability condition and the concept of discrete stability. We review the classical results of Babuśka, Mikhlin and Brezzi, and finish the exposition with fundamentals of the Discontinuous Petrov Galerkin (DPG) method. The week-long course consists of three 1.5 hour lectures per day accompanied with a Q/Q session afterwards.

[10, 6, 2, 9, 5, 8, 7, 1, 3, 4]

Day 1

1. Classical calculus of variations. Concept of a variational formulation.
2. Abstract variational problem.
3. Diffusion-convection-reaction model problem. Different variational formulations.
4. Distributional derivatives and different energy spaces.
5. Bubnov- and Petrov-Galerkin methods.

Day 2

1. Babuška - Nečas and Banach Closed Range Theorems.
2. Riesz Representation and Lax Milgram Theorems.
3. Babuška Theorem and concept of discrete stability.
4. Ritz method.
5. Exact sequence elements.

Day 3

1. Examples of coercive problems.
2. Compact perturbations of coercive problems. Mikchlin's theory of asymptotic stability.
3. Mixed problems, Brezzi's theory.

Day 4

1. The ideal PG method with optimal test functions. Equivalent formulations - residual minimization and the mixed problem
2. Variational formulations with broken (discontinuous) test functions.
3. The practical DPG method.

Day 5

1. Duality pairings and concept of optimal test norm.
2. Multigrid method for DPG.
3. Double adaptivity.

References

[1] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan. Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Comput. Math. Appl.*, 72(3):494–522, 2016.

[2] L. Demkowicz. Various variational formulations and Closed Range Theorem. Technical report, ICES, January 15–03.

[3] L. Demkowicz. Asymptotic convergence in finite and boundary element methods. Part 1: Theoretical results. *Comput. Math. Appl.*, 27(12):69–84, 1994.

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[8] L. Demkowicz, J. Kurtz, D. Pardo, M. Paszyński, W. Rachowicz, and A. Zdunek. *Computing with hp Finite Elements. II. Frontiers: Three-Dimensional Elliptic and Maxwell Problems with Applications*. Chapman & Hall/CRC, October 2007.

[9] F. Fuentes, B. Keith, L. Demkowicz, and S. Nagaraj. Orientation embedded high order shape functions for the exact sequence elements of all shapes. *Comput. Math. Appl.*, 70:353–458, 2015.

[10] J.T. Oden and L.F. Demkowicz. *Applied Functional Analysis for Science and Engineering*. Chapman & Hall/CRC Press, Boca Raton, 2018. Third edition.

You may also consult notes on my web page:

`http://users.ices.utexas.edu/~leszek/classes.html`

(Lecture Notes for Advanced Theory of Finite Element Methods (EM394H/CAM394H)).

Exercises

1. Recall the definitions of inner product (preHilbert) and normed vector spaces. Prove that every inner product $(u, v)_V$ generates the corresponding *Eucklidean norm* given by the formula:

$$\|v\|_V^2 := (v, v)_V \, .$$

   Hence, any preHilbert space is automatically a normed space. *Hint:* You will have to prove the abstract Cauchy-Schwarz inequality:

$$|(u, v)_V\| \le \|u\|_V \, \|v\|_V , \quad u, v, \in V$$

   Consult other sources if necessary. (5 points)

2. Recall the definition of a metric space. Prove that every norm generates a metric given by:

$$d(u, v) := \|u - v\|_V \, .$$

   Hence, every normed space is automatically a metric space. (5 points)

3. Derive the variational formulation and the corresponding Euler-Lagrange boundary-value problem for the two-dimensional minimization problem:

$$\begin{cases} u = u_0 \text{ on } \Gamma_1 \\ \displaystyle\int_\Omega F(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y))\; dxdy \to \min \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded two-dimensional domain with boundary Γ split into two disjoint parts, $\Gamma = \Gamma_1 \cup \Gamma_2$. (5 points)

4. (An interface problem) Consider the elastic beam pictured in Fig. 1. Deflection $w(x)$ of the beam minimizes the *total potential energy* given by the functional

$$J(w) = \frac{1}{2}\int_0^{3l/2} EI(w'')^2 - \left[\int_0^{3l/2} qw + P_0 w(\frac{3l}{2}) + M_0 w'(\frac{3l}{2})\right]$$

among all possible displacements that satisfy the *kinematic BC*:

$$w(0) = w'(0) = w(l) = 0$$

- Derive the Gâteaux derivative of cost functional $J(w)$ and the corresponding variational formulation for the problem.
- Use integration by parts (twice) and the Fourier's Lemma argument to derive the corresponding E-L equation(s) in subintervals $(0, l)$ and $l, 3l/2)$, boundary conditions at $x = 3l/2$ and interface conditions at $x = l$.
- Show the (formal) equivalence between the variational formulation and the E-L interface boundary-value problem.

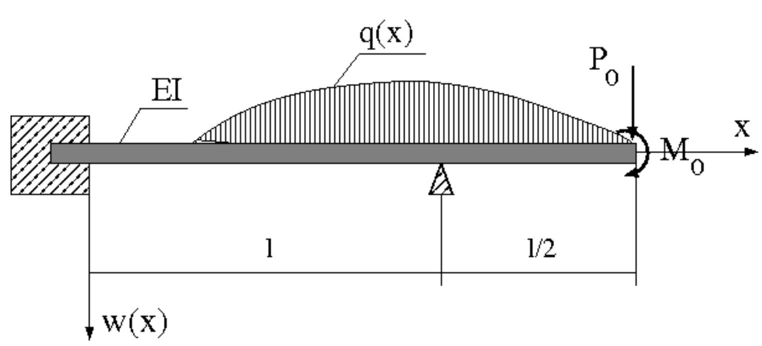

Figure 1: An elastic beam example

(10 points)

5. Integration by parts formulas. Let $\Omega \subset I\!\!R^3$ be a domain with boundary $\partial\Omega$. Use elementary integration by parts formula to derive the following integration by parts formulas.

$$\int_\Omega \nabla u \, v = -\int_\Omega u \, \nabla v + \int_{\partial\Omega} n u \, v$$
$$\int_\Omega (\nabla \times E) \cdot F = \int_\Omega E \cdot (\nabla \times F) + \int_{\partial\Omega} (n \times E) \cdot F$$
$$\int_\Omega (\nabla \cdot u) \, v = -\int_\Omega u \cdot (\nabla v) + \int_{\partial\Omega} u \cdot n \, v$$

(5 points)

6. Consider the diffusion-convection-reaction problem:

$$-\text{div}\, \underbrace{(a\nabla u - bu)}_{=:\sigma} + cu = f$$

accompanied with BC's:

$$u = 0 \text{ on } \Gamma_u \qquad \text{and} \qquad \sigma \cdot n = 0 \text{ on } \Gamma_\sigma \,,$$

where Γ_u, Γ_σ are two disjoint parts of boundary $\Gamma = \partial\Omega$. Replace the second order diffusion-convection-reaction equation with a system of first order equations. Derive then (formally, no math details expected) the corresponding *six* variational formulations and identify the corresponding functional settings. Identify the (group) unknown, the (group) test function, and the bilinear and linear forms. Explain what we mean by a *symmetric functional setting* ? (10 points)

7. Consider the second order problem and attempt to derive the classical variational formulation (Reduced Formulation I) *directly*. Identify the corresponding functional setting: energy trial and test space, bilinear and linear forms. (5 points)

8. Equivalence of continuity and boundedness for linear(antilinear) forms. Let V be a normed vector space and l be a linear (antilinear) functional defined on V. Prove that the following conditions are equivalent to each other. (5 points)

   (i) l is continuous on V,

   (ii) l is continuous at 0 (zero vector),

   (iii) l is *bounded*, i.e. there exists $C > 0$ such that

   $$|l(v)| \leq C\|v\|_V$$

   where $\|v\|_V$ is the norm in V.

9. Equivalence of continuity and boundedness for bilinear(sesquilinear) forms. Let U, V be normed vector spaces and b be a bilinear (sesquilinear) functional defined on $U \times V$. Prove that the following conditions are equivalent to each other. (5 points)

(i) b is continuous on $U \times V$,

(ii) b is continuous at $(0,0)$,

(iii) b is *bounded*, i.e. there exists $M > 0$ such that

$$|b(u,v)| \leq M \|u\|_U \, \|v\|_V \, .$$

10. Dual norm. Let V be a normed vector space and l be a continuous (bounded) linear (antilinear) functional defined on V. Let $\|l\|$ be the "smallest" constant that we can use in the boundedness condition,

$$\|l\| := \inf\{C \, : \, |l(v)| \leq C\|v\|_V\}$$

(a) Prove equivalent characterizations for $\|l\|$,

$$\|l\| = \sup_{v \neq 0} \frac{|l(v)|}{\|v\|} = \sup_{\|v\|=1} |l(v)|$$

(b) Let V' be the collection of all bounded linear (antilinear) functionals defined on V. Argue that V' is close wrt the standard operations on functions and, therefore, constitutes a subspace of algebraic dual V^* consisting of all linear (antilinear) functionals on V. Prove that $\|l\|$ satisfies the axioms for a norm, i.e V' is a normed space (called the *topological dual* of space normed space V).

(10 points)

11. Distributional derivatives. Let a domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be split into two subdomains Ω_1, Ω_2 with a smooth interface Γ. Let u, E, v be functions consisting of two smooth branches u^I, E^I, v^I, $I = 1, 2$ defined in the subdomains. By "smooth" we understand $u^I \in C^1(\overline{\Omega_I})$ etc. Let n be the unit vector on interface Γ pointing from subdomain Ω_1 into subdomain Ω_2.

(i) Let $\phi \in C_0^\infty(\Omega)$ be a Schwartz test function (scalar- or vector-valued). Use elementary integration by parts to derive the following formulas:

$$-\int_\Omega u \nabla \phi = \sum_I \int_{\Omega_I} \nabla u^I \, \phi + \int_\Gamma [u] n \phi \, ,$$

$$\int_\Omega E \, \nabla \times \phi \, , = \sum_I \int_{\Omega_I} \nabla \times E^I \, \phi + \int_\Gamma [n \times E] \phi \, ,$$

$$\int_\Omega v \, \nabla \cdot \phi = \sum_I \int_{\Omega_I} \nabla \cdot v^I \, \phi + \int_\Gamma [n \cdot v] \phi$$

where

$$[u] = u^2 - u^1, \quad [n \times E] = n \times (E^2 - E^1), \quad [n \cdot v] = n \cdot (v^2 - v^1) \, .$$

(ii) Interpret the formulas above in the language of distributions using the definition of regular distributions, distributional derivatives and corresponding operators of grad, curl and div understood in the distributional sense. You will have to introduce a multidimenional equivalent of Dirac's delta.

(iii) Conclude that functions u, E, v belong to energy spaces $H^1(\Omega), H(\text{curl}, \Omega), H(\text{div}, \Omega)$ *if and only if* the corresponding continuity conditions across the interface Γ are satisfied:

$$[u] = 0, \quad [n \times E] = 0, \quad [n \cdot v] = 0\,.$$

(20 points)

12. Mikhlin Theorem (for those of you that are mathematically inclined). Use my lecture notes or [3] to reproduce the proof of Mikhlin Theorem. (10 points)

13. Interpretation of inf-sup constant in terms of eigenvalues. Consider a baby *vibrations of an elastic bar problem* where

$$b(u,v) = \int_0^1 u'\bar{v}'\,dx - \omega^2 \int_0^1 uv\,dx$$

with $u, v \in H_0^1(0,1)$ with

$$H_0^1(0,1) := \{u \in H^1(0,1)\,:\, u(0) = u(1) = 0\}\,.$$

Introduce the eigenvalue problem for the 1D Laplacian:

$$\begin{cases} u \in H_0^1(0,1) \\ \int_0^1 u'v'\,dx = \lambda \int_0^1 uv\,dx \quad v \in H_0^1(0,1)\,. \end{cases}$$

Compute the eigenvalues λ_i, $i = 1, 2, \infty$ and then compute the inf-sup constant γ in terms of the eigenvalues λ_i and frequency ω. Consult my lecture notes or [3] for help. Reproduce the same argument for the discrete case and argue that

$$\gamma_h \to \gamma \quad \text{if} \quad \lambda_{i,h} \to \lambda_i\,.$$

(20 points)

14. Second Brezzi condition (for those of you mathematically inclined). Finish the argument from the class and use my lecture notes to derive the second Brezzi (inf-sup in kernel) condition from the Babuška inf-sup condition.

(10 points)

15. Affine coordinates. Prove the following facts about the affine coordinates:

- The affine coordinates are independent of the enumeration of vertices (in the presented construction, we considered vectors $x - a_0, a_i - a_0$, $i = 1, 2, 3$, so it looks like things might depend upon the choice of vertex a_0).
- The *affine coordinates are invariant under affine transformations*: if λ_i are affine coordinates of a point x with respect to vertices a_i then λ_i are also affine coordinates of a point Tx with respect to vertices Ta_i, for any bijective affine map T:

$$Tx := a + Ax$$

  where $a \in \mathbb{R}^3$, and A is a non-singular 3×3 matrix.
- In 2D, the affine coordinates may be interpreted as *area coordinates*. Prove that

$$\lambda_i = \frac{\text{area of } T_i}{\text{area of } T}, \quad i = 0, 1, 2$$

  where subtriangles T_i of triangle T are defined in Fig. 2.

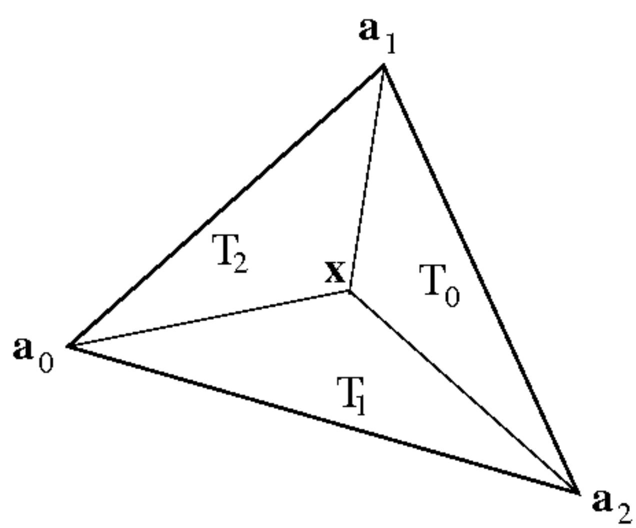

Figure 2: Area coordinates.

Be concise. (10 points)

16. Whitney shape functions. Write down formulas for the Whitney shape functions in terms of affine coordinates and their gradients. Discuss their vanishing properties and explain how you "glue" them to obtain Galerkin basis functions for the energy spaces forming the exact sequence. Use [9] if necessary.

(10 points)

17. Shape functions for the lowest order hexahedron. Write out shape functions for the lowest order hexahedron in terms of 1D affine coordinates and their derivatives. Use [9] if necessary.

    (10 points)